Multivariable Analysis Lecture Notes (2024/2025)

Griffin Reimerink

Contents

0.1 Recap of vector spaces

Definition *Vector space*

A **vector space** V is a set of vectors that is closed under vector addition and scalar multiplication.
The vector space equipped with the addition $(+)$ and multiplication $(\cdot)$ operations satisfies the following rules:

1. There exists an element $\mathbf{0}$ in V such that $\mathbf{0} + u = u$ for all $u \in V$.
2. Vector addition is commutative: $u + v = v + u$
3. Vector addition is associative: $u + (v + w) = (u + v) + w$
4. For each $u \in V$, there exists $-u \in V$ such that $u + (-u) = \mathbf{0}$
5. For all $u \in V$, $1 \cdot u = u$
6. For all $a, b \in \mathbb{F}$ and all $u \in V$, $a \cdot (b \cdot u) = (ab) \cdot u$
7. For all $c \in \mathbb{F}$ and all $u, v \in V$, $c \cdot (u + v) = (c \cdot u) + (c \cdot v)$
8. For all $a, b \in \mathbb{F}$ and all $u \in V$, $(a + b) \cdot u = (a \cdot u) + (b \cdot u)$

Definition *Linear transformation*

A **linear transformation** is a map $T : V \to W$ between vector spaces satisfying

$$T(a\boldsymbol{v}_1 + b\boldsymbol{v}_2) = aT(\boldsymbol{v}_1) + bT(\boldsymbol{v}_2) \qquad \text{for all} \quad \boldsymbol{v_1, v_2} \in V \quad a, b \in \mathbb{R}$$

Definition *Dimension*

If a vector space V has a finite basis, then all of its bases are finite and contain the same number of elements.
The number of elements in a basis of V called the **dimension** of V.

Change of basis

Let V, W be vector spaces. Let v, v' be bases of V and let w, w' be bases of W.
We denote the **change of basis matrix** from v to v' by $P_{v \to v'}$. Multiplying the representation of a vector in basis v by this matrix yields the same vector's representation in the basis v'.
The i-th column of this matrix is the representation in basis v of the i-th basis vector of v'.

Let $T : V \to W$ be a linear transformation. Then the **change of basis formula** is given by:

$$T_{v',w'} = P^{-1}_{w' \to w} T_{v,w} P_{v' \to v}$$

1 Differentiation

1.1 Definitions

Definition *Partial derivative*

Let $U \subseteq \mathbb{R}^n$ be open and let $\boldsymbol{f} = (f_1, \dots, f_m)$ be a map $U \to \mathbb{R}^m$.
The **partial derivative** of $\boldsymbol{f}$ w.r.t. the i-th variable evaluated at $\boldsymbol{a} \in U$ is:

$$D_i \boldsymbol{f}(\boldsymbol{a}) := \lim_{h \to 0} \frac{\boldsymbol{f}(a_1, \dots, a_i + h, \dots, a_n) - \boldsymbol{f}(a_1, \dots, a_i, \dots, a_n))}{h} = \begin{bmatrix} D_i f_1(\boldsymbol{a}) \\ \dots \\ D_i f_m(\boldsymbol{a}) \end{bmatrix}$$

whenever such a limit exists.

Definition *Jacobian matrix*

Let $U \subseteq \mathbb{R}^n$ be open. The **Jacobian matrix** of $\boldsymbol{f} : U \to \mathbb{R}^m$ is the $m \times n$ matrix

$$J\boldsymbol{f}(\boldsymbol{a}) := \begin{bmatrix} D_1 f_1(\boldsymbol{a}) & \dots & D_n f_1(\boldsymbol{a}) \\ \vdots & \ddots & \vdots \\ D_1 f_m(\boldsymbol{a}) & \dots & D_n f_m(\boldsymbol{a}) \end{bmatrix}$$

Definition *Derivative*

Let $U \subseteq \mathbb{R}^n$ be open and let $\boldsymbol{f} : U \to \mathbb{R}^m$ be a map.
$\boldsymbol{f}$ is **differentiable** at $\boldsymbol{a} \in U$ if there exists a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ such that:

$$\lim_{\boldsymbol{h} \to 0} \frac{\boldsymbol{f}(\boldsymbol{a} + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{a}) - L(\boldsymbol{h})}{|\boldsymbol{h}|} = \boldsymbol{0} \in \mathbb{R}^m$$

This linear transformation L is the **derivative** of $\boldsymbol{f}$ at $\boldsymbol{a}$, denoted by $D\boldsymbol{f}(\boldsymbol{a})$.

Theorem

If $\boldsymbol{f}$ is differentiable at $\boldsymbol{a}$, then:

- $\boldsymbol{f}$ is continuous at $\boldsymbol{a}$.
- All partial derivatives of $\boldsymbol{f}$ at $\boldsymbol{a}$ exist.
- $J\boldsymbol{f}(\boldsymbol{a})$ is the matrix representation of $D\boldsymbol{f}(\boldsymbol{a})$.

Definition *Directional derivative*

Let $U \subseteq \mathbb{R}^n$ be open and let $\boldsymbol{f} : U \to \mathbb{R}^m$ be a map.
The **directional derivative** of $\boldsymbol{f}$ at $\boldsymbol{a} \in U$ in the direction $\boldsymbol{v}$ is

$$\lim_{h \to 0} \frac{\boldsymbol{f}(\boldsymbol{a} + h\boldsymbol{v}) - \boldsymbol{f}(\boldsymbol{a})}{h}$$

Proposition

If $\boldsymbol{f} : U \to \mathbb{R}^m$ is differentiable at $\boldsymbol{a} \in U$, then all directional derivatives of $\boldsymbol{f}$ at $\boldsymbol{a}$ exist, and the directional derivative in the direction v is given by $D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{v}$

Definition *Function of class C^p*

A function $f : U \to \mathbb{R}$ is of **class** C^p if all partial derivatives up to order p exist and are continuous on U.
A function $\boldsymbol{f} = (f_1, \dots, f_m)$ is of class C^p if $f_1, \dots, f_m$ are all of class C^p.

Theorem

Let $\boldsymbol{f} : U \to \mathbb{R}^m$ be a C^1-mapping. Then $\boldsymbol{f}$ is differentiable on U and its derivative is given by its Jacobian.

1.2 Differentiation rules

Proposition

If $f(A) = A^{-1}$ is defined om the set of invertible matrices, then $Df(A)H = -A^{-1}HA^{-1}$

Theorem

1. If $\boldsymbol{f} : U \to \mathbb{R}^m$ is constant, then its derivative is the zero map.
2. If $\boldsymbol{f} : U \to \mathbb{R}^m$ is linear, then it is differentiable everywhere, and its derivative at all points is $\boldsymbol{f}$ itself.
3. If $f_1, \ldots, f_m : U \to \mathbb{R}$ are differentiable at $\boldsymbol{a}$, then $\boldsymbol{f} = (f_1, \ldots, f_m)$ is differentiable at $\boldsymbol{a}$ with derivative
$$(D\boldsymbol{f}(\boldsymbol{a}))\boldsymbol{v} = \begin{bmatrix} Df_1(\boldsymbol{a})\boldsymbol{v} \\ \vdots \\ Df_m(\boldsymbol{a})\boldsymbol{v} \end{bmatrix}$$
4. If $\boldsymbol{f}, \boldsymbol{g} : U \to \mathbb{R}^m$ are differentiable at $\boldsymbol{a}$, then so is $\boldsymbol{f} + \boldsymbol{g}$, with derivative
$$D(\boldsymbol{f} + \boldsymbol{g})(\boldsymbol{a}) = D\boldsymbol{f}(\boldsymbol{a}) + D\boldsymbol{g}(\boldsymbol{a})$$
5. If $f : U \to \mathbb{R}$ and $\boldsymbol{g} : U \to \mathbb{R}^n$ are differentiable at $\boldsymbol{a}$, then so is $f\boldsymbol{g}$, with derivative
$$(D(f\boldsymbol{g})(\boldsymbol{a}))\boldsymbol{v} = f(\boldsymbol{a})(D\boldsymbol{g}(\boldsymbol{a}))\boldsymbol{v} + (Df(\boldsymbol{a})\boldsymbol{v})\boldsymbol{g}(\boldsymbol{a})$$
6. If $f : U \to \mathbb{R}$ and $\boldsymbol{g} : U \to \mathbb{R}^n$ are differentiable at $\boldsymbol{a}$ and $f(\boldsymbol{a}) \neq 0$, then so is $\frac{\boldsymbol{g}}{\boldsymbol{f}}$, with derivative
$$\left(D\left(\frac{\boldsymbol{g}}{f}\right)(\boldsymbol{a})\right)\boldsymbol{v} = \frac{D\boldsymbol{g}(\boldsymbol{a}))\boldsymbol{v}}{f(\boldsymbol{a})} - \frac{((Df(\boldsymbol{a}))\boldsymbol{v})(\boldsymbol{g}(\boldsymbol{a}))}{(f(\boldsymbol{a}))^2}$$
7. If $\boldsymbol{f}, \boldsymbol{g} : U \to \mathbb{R}^m$ are differentiable at $\boldsymbol{a}$, then so is $\boldsymbol{f} \cdot \boldsymbol{g}$, with derivative
$$(D(\boldsymbol{f} \cdot \boldsymbol{g})(\boldsymbol{a}))\boldsymbol{v} = (D\boldsymbol{f}(\boldsymbol{a}))\boldsymbol{v} \cdot \boldsymbol{g}(\boldsymbol{a}) + \boldsymbol{f}(\boldsymbol{a}) \cdot (D\boldsymbol{g}(\boldsymbol{a}))\boldsymbol{v}$$

Theorem *Chain rule*

Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open, let $\boldsymbol{g} : U \to V$ and $\boldsymbol{f} : V \to \mathbb{R}^p$ be maps, and $\boldsymbol{a} \in U$.
If $\boldsymbol{g}$ is differentiable at $\boldsymbol{a}$ and $\boldsymbol{f}$ is differentiable at $\boldsymbol{g}(a)$, then $\boldsymbol{f} \circ \boldsymbol{g}$ is differentiable at $\boldsymbol{a}$, with derivative:

$$D(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{a})) = (D\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{a}))) \circ (D\boldsymbol{g}(\boldsymbol{a})) = J\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{a}))J\boldsymbol{g}(\boldsymbol{a})$$

Theorem *Mean value theorem*

Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be differentiable.
Let the segment $[\boldsymbol{a}, \boldsymbol{b}]$ (the image of $t \mapsto (1-t)\boldsymbol{a} + t\boldsymbol{b}$, $0 \leq t \leq 1$) be contained in U.
Then there exists $\boldsymbol{c} \in [\boldsymbol{a}, \boldsymbol{b}]$ such that

$$f(\boldsymbol{b}) - f(\boldsymbol{a}) = (Df(\boldsymbol{c}))(\boldsymbol{b} - \boldsymbol{a})$$

1.3 Newton method

Definition *Newton method*

Let $U \subseteq \mathbb{R}^n$ be open and let $\boldsymbol{f} : U \to \mathbb{R}^m$ be differentiable on U. The equation $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$ can be solved using the **Newton method**, which starts with an initial guess $\boldsymbol{x}_0$, and iteratively solves the sequence of equations:

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i - (D\boldsymbol{f}(\boldsymbol{x}_i))^{-1}\boldsymbol{f}(\boldsymbol{x}_i) \qquad \text{or alternatively: } (D\boldsymbol{f}(\boldsymbol{x}_i))(\boldsymbol{x}_{i+1} - \boldsymbol{x}_i) = -\boldsymbol{f}(\boldsymbol{x}_i)$$

Definition *Frobenius norm*

$$|A| = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j}^2\right)^{\frac{1}{2}}$$

Definition *Lipschitz condition*

Let $U \subseteq \mathbb{R}^n$ be open $\boldsymbol{f} : U \to \mathbb{R}^m$ differentiable.
The derivative $(D\boldsymbol{f}(\boldsymbol{x}))$ satisfies a **Lipschitz condition** on $V \subseteq U$ with **Lipschitz ratio** M if:

$$|(D\boldsymbol{f}(\boldsymbol{x})) - (D\boldsymbol{f}(\boldsymbol{y}))| \leq M|\boldsymbol{x} - \boldsymbol{y}| \qquad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in V$$

Definition *Second partial derivative*

Let $U \subseteq \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}$ differentiable. If the function $D_i f$ is itself differentiable, then its partial derivative with respect to the j-th variable $D_j(D_i f)$ is called a **second partial derivative** of f.

Proposition

Let $U \subseteq \mathbb{R}^n$ be open and let $\boldsymbol{f} : U \to \mathbb{R}^n$ be C^2.
If $|D_k D_j f_i(x)| \leq c_{i,j,k}$ for any $x \in U$ and any triple indices $1 \leq i, j, k \leq n$, then

$$\text{for all } u, v \in U \qquad |D\boldsymbol{f}(u) - D\boldsymbol{f}(v)| \leq \left(\sum_{1 \leq i,j,k \leq n} (c_{i,j,k})^2\right)^{\frac{1}{2}} |u - v|.$$

Theorem *Kantorovich's theorem*

Let $\boldsymbol{x}_0 \in \mathbb{R}^n$, U an open neighborhood of $\boldsymbol{x}_0$ in $\mathbb{R}^n$, and $\boldsymbol{f} : U \to \mathbb{R}^n$ differentiable with its derivative $D\boldsymbol{f}(\boldsymbol{x}_0)$ invertible. Define

$$\boldsymbol{h}_0 = -(D\boldsymbol{f}(\boldsymbol{x}_0))^{-1}\boldsymbol{f}(\boldsymbol{x}_0) \qquad \boldsymbol{x}_1 = \boldsymbol{x}_0 + \boldsymbol{h}_0 \qquad U_1 = B_{|\boldsymbol{h}_0|}(\boldsymbol{x}_1)$$

If $\overline{U}_1 \subset U$ and the derivative $D\boldsymbol{f}(\boldsymbol{x})$ satisfies the Lipschitz condition:

$$|D\boldsymbol{f}(\boldsymbol{u}_1) - D\boldsymbol{f}(\boldsymbol{u}_2)| \leq M|\boldsymbol{u}_1 - \boldsymbol{u}_2| \qquad \text{for all } \boldsymbol{u}_1, \boldsymbol{u}_2 \in \overline{U}_1$$

and the following inequality is satisfied:

$$|\boldsymbol{f}(\boldsymbol{x}_0)||D\boldsymbol{f}(\boldsymbol{x}_0)^{-1}|^2 M \leq \frac{1}{2}$$

then the equation $\boldsymbol{f}(x) = \boldsymbol{0}$ has a unique solution in the closed ball $\overline{\boldsymbol{U}}_1$,
and the Newton method converges to it with initial guess $\boldsymbol{x}_0$.

1.4 Inverse and implicit functions

Definition *Strictly monotone function*

A function is **strictly monotone** if either $x < y \implies f(x) < f(y)$ or $x < y \implies f(x) > f(y)$.

Theorem *Inverse function theorem* $(\mathbb{R})$

Let $f : [a, b] \to [c, d]$ be continuous with $f(a) = c$, $f(b) = d$ and f strictly monotone on $[a, b]$. Then

1. There exists a unique continuous function $g : [c, d] \to [a, b]$ such that

$$f(g(y)) = y \quad \forall y \in [c, d] \qquad\qquad g(f(x)) = y \quad \forall x \in [a, b]$$

2. One can find $g(y)$ by solving $y - f(x) = 0$ for x using the bisection method.

3. If f is differentiable at $x \in (a, b)$ and $f'(x) \neq 0$, then g is differentiable at $f(x)$ and $g(f'(x)) = \dfrac{1}{f'(x)}$

Theorem *Inverse function theorem* ($\mathbb{R}^n$)

If a mapping $\mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, and its derivative is invertible at some point $\boldsymbol{x}_0$, then $\boldsymbol{f}$ is locally invertible, with differentiable inverse, in some neighborhood of $\boldsymbol{f}(\boldsymbol{x}_0)$.

Definition *Implicit function*

An **implicit function** is a function that is defined by an implicit relation of the form $f(x_1, \ldots, x_n) = 0$.

Theorem *Implicit function theorem*

Let $U \in \mathbb{R}^n$ be open and $c \in U$. Let $\boldsymbol{f} : U \to \mathbb{R}^{n-k}$ be a C^1-mapping such that $\boldsymbol{f}(c) = 0$ and $D\boldsymbol{f}(\boldsymbol{c})$ is surjective. Then the system of linear equations $(D\boldsymbol{f}(\boldsymbol{c}))(\boldsymbol{x}) = 0$ has $n-k$ **passive variables** and k **active variables**, and there exists a neighborhood of $\boldsymbol{c}$ in which $\boldsymbol{f} = 0$ implicitly defines the $n-k$ passive variables as a function $\boldsymbol{g}$ of the k active variables. This function $\boldsymbol{g}$ is called an **implicit function**.

* The full versions of the inverse and implicit function theorem can be found in the slides of Lecture 3.

1.5 Manifolds

1.5.1 Smooth manifolds

Definition *Graph*

The **graph** $\Gamma(\boldsymbol{f})$ of a function $\boldsymbol{f} : \mathbb{R}^k \to \mathbb{R}^{n-k}$ is the set of points $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{y}$.

Definition *Smooth manifold*

A subset $M \subset \mathbb{R}^n$ is a **smooth k-dimensional manifold** if locally it is the graph of a C^1-mapping $\boldsymbol{f} : \mathbb{R}^k \to \mathbb{R}^{n-k}$
By "locally" we mean that for all $\boldsymbol{x} \in M$, there exists some neighborhood U such that $M \cap U$ is the graph of a mapping expressing $n-k$ coordinates as a function of the other k.
If M is locally the graph of a C^k-mapping, we call it a **C^k-manifold**.
1-dimensional manifolds are called **smooth curves** and 2-dimensional manifolds are called **smooth surfaces**.

Theorem *Embedded manifold theorem*

Let $U \in \mathbb{R}^n$ be open and let $\boldsymbol{f} : U \to \mathbb{R}^{n-k}$ be C^1. Let $M \subset \mathbb{R}^n$ such that $M \cap U = \{\boldsymbol{z} \in U \mid \boldsymbol{f}(\boldsymbol{z}) = 0\}$.
If $D\boldsymbol{f}(\boldsymbol{z})$ is surjective for every $\boldsymbol{z} \in M \cap U$, then $M \cap U$ is a **smooth k-dimensional manifold embedded in $\mathbb{R}^n$**.
If every $\boldsymbol{z} \in M$ is in such a U, then M is a k-dimensional manifold.

Conversely, if M is a smooth k-dimensional manifold embedded in $\mathbb{R}^n$, then every $\boldsymbol{z} \in M$ has a neighborhood $U \in \mathbb{R}^n$ such that there exists a C^1-mapping $\boldsymbol{f} : U \to \mathbb{R}^{n-k}$ with $D\boldsymbol{f}(\boldsymbol{z})$ surjective and $M \cap U = \{\boldsymbol{y} \mid \boldsymbol{f}(\boldsymbol{y}) = 0\}$

We call $\{\boldsymbol{x} \in U \mid \boldsymbol{f}(\boldsymbol{x}) = 0\}$ the **locus** of $\boldsymbol{f}$.

Theorem

Let $M \subset \mathbb{R}^m$ be a k-dimensional manifold, U an open subset of $\mathbb{R}^n$,
and $\boldsymbol{f} : U \to \mathbb{R}^m$ a C^1-mapping whose derivative is surjective at every point $\boldsymbol{x} \in \boldsymbol{f}^{-1}(M)$.
Then the inverse image $\boldsymbol{f}^{-1}(M)$ is a submanifold of $\mathbb{R}^n$ of dimension $k + n - m$.

Definition *Parametrization*

A **parametrization** of a k-dimensional manifold $M \subseteq \mathbb{R}^n$ is a mapping $\gamma : U \subseteq \mathbb{R}^k \to M$ satisfying the following conditions:

1. U is open
2. γ is C^1 and bijective
3. $D_\gamma(u)$ is surjective for all $u \in U$

1.5.2 Tangent spaces

Definition *Tangent space*

Let M be a k-dimensional manifold. The **tangent space** to M at $\boldsymbol{z}_0 = (\boldsymbol{x}_0, \boldsymbol{y}_0)$, denoted by $T_{\boldsymbol{z}_0}M$, is the graph of the linear transformation $D\boldsymbol{f}(\boldsymbol{x}_0)$.

Theorem

Let $U \subseteq \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^{n-k}$ a C^1-mapping. If $F(\boldsymbol{z}) = 0$ describes a manifold M and $DF(\boldsymbol{z}_0)$ is surjective for some $\boldsymbol{z}_0 \in M$, then the tangent space $T_{\boldsymbol{z}_0}M$ is the kernel of $DF(\boldsymbol{z}_0)$.

Proposition

Let $U \subseteq \mathbb{R}^k$ be open and let γ be a parametrization of a manifold M. Then $T_{\gamma(u)}M = \operatorname{im}(D_\gamma(u))$

1.5.3 Differentiable maps on manifolds

Definition *Maps of class C^p on manifolds*

Let $M \subseteq \mathbb{R}^n$ be an m-dimensional manifold, and $\boldsymbol{f} : M \to \mathbb{R}^k$ a map. The map $\boldsymbol{f}$ is of **class** C^p if every $\boldsymbol{x} \in M$ has a neighborhood $U \subseteq \mathbb{R}^n$ such that there exists a map $\widetilde{\boldsymbol{f}} : U \to \mathbb{R}^k$ of class C^p with $\boldsymbol{f}|U \cap M = \widetilde{\boldsymbol{f}}|U \cap M$

Proposition

If $p \geq 1$ (with p and $\widetilde{\boldsymbol{f}}$ as in the definition above), then

$$D\boldsymbol{f}(x) : T_xM \to \mathbb{R}^k := D\widetilde{\boldsymbol{f}}(x)|_{T_xM}$$

does not depend on the choice of $\widetilde{\boldsymbol{f}}$.

Proposition

Let $M \subseteq \mathbb{R}^n$ be an n-dimensional manifold and $\boldsymbol{f} : M \to \mathbb{R}^k$ be a C^1-map. Let $P \subseteq M$ be the set where $\boldsymbol{f} = 0$. If $D\boldsymbol{f}(\boldsymbol{x}) : T_xM \to \mathbb{R}^k$ is onto at every $\boldsymbol{x} \in P$, then P is an $(m-k)$-dimensional manifold.

Proposition *Chain rule on manifolds*

Let $M \subseteq \mathbb{R}^n$ be a manifold, let $U \subseteq \mathbb{R}^\ell$ be open, and let $\boldsymbol{f} : M \to \mathbb{R}^k$ and $\boldsymbol{g} : U \to M$ be C^1 maps. Then:

$$D(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = D\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))D\boldsymbol{g}(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in U$$

1.6 Taylor polynomials

1.6.1 Multiexponents and polynomials

Definition *Multiexponent*

- A **multiexponent** I is an ordered finite list of nonnegative integers $I = (i_1, \ldots, i_n)$
- The set of multiexponents with n entries is denoted by $\mathcal{I}_n$
- For any $I \in \mathcal{I}_n$, the **total degree** of I is $\deg I := \sum_{j=i}^{n} i_j$ and the **factorial** is $I! := i_1! \cdots i_n!$
- By $\mathcal{I}_n^k$ we denote the set of multiexponents with n entries of total degree k.
- For any $I \in \mathcal{I}_n$, $\boldsymbol{x}^I := x_1^{i_1} \cdots x_n^{i_n}$ and $D_I f := D_1^{i_1} D_2^{i_2} \cdots D_n^{i_n} f$

General form of a degree m polynomial in n variables

$$p(\boldsymbol{x}) = \sum_{k=0}^{m} \sum_{I \in \mathcal{I}_n^k} a_I \boldsymbol{x}^I$$

Theorem *Equality of crossed partials*

Let $U \in \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ a function such that all of its partial derivatives $D_i f$ exist and are themselves differentiable at $a \in U$. Then for every pair of variables x_i, x_j, $D_j(D_i f)(\boldsymbol{a}) = D_i(D_j f)(\boldsymbol{a})$

Corollary

If $f : U \to \mathbb{R}$ is a function all of whose partial derivatives up to order k are continuous, then the partial derivatives of order up to k do not depend on the order in which they are computed.

Proposition *Coefficients in terms of partial derivatives*

Let $p(\boldsymbol{x}) = \sum_{k=0}^{m} \sum_{J \in \mathcal{I}_n^k}$ Then for any $I \in \mathcal{I}_n$, we have $a_I = \dfrac{D_I p(0)}{I!}$

1.6.2 Taylor polynomials

Theorem *Taylor polynomials in 1 dimension*

If $U \subseteq \mathbb{R}$ is an open subset and $f : U \to \mathbb{R}$ is k-times differentiable on U, then the polynomial

$$p_{f,a}^k(a+h) := f(a) + f'(a)h + \sum_{j=2}^{k} \frac{f^{(j)}(a)}{j!} h^j$$

is called the **Taylor polynomial** of degree k, and it is the best approximation of f at a:

$$\lim_{h \to 0} \frac{f(a+h) - p_{f,a}^k(a+h)}{h^k} = 0$$

Definition *Taylor polynomials in higher dimension*

Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be a C^k function. Then the polynomial of degree k

$$P_{f,\boldsymbol{a}}^k(\boldsymbol{a}+\boldsymbol{h}) = \sum_{m=0}^{k} \sum_{I \in \mathcal{I}_n^m} \frac{1}{I!} D_I f(\boldsymbol{a}) \boldsymbol{h}^I$$

is called the **Taylor polynomial** of degree k of f at $\boldsymbol{a}$.

If $\boldsymbol{f} : U \to \mathbb{R}^n$ is a C^k function, then its Taylor polynomial is the polynomial map $U \to \mathbb{R}^n$ whose coordinates are the Taylor polynomials of the coordinate functions of $\boldsymbol{f}$.

Theorem *Uniqueness of the Taylor polynomial*

Let $U \subseteq \mathbb{R}^n$ be open, $\boldsymbol{a} \in U$ and $f : U \to \mathbb{R}$ a C^k function.

1. The $P_{f,\boldsymbol{a}}^k(\boldsymbol{a}+\boldsymbol{h})$ is the unique polynomial of degree k with the same partial derivative up to order k as f.
2. The polynomial $P_{f,\boldsymbol{a}}^k(\boldsymbol{a}+\boldsymbol{h})$ best approximates f near $\boldsymbol{a}$ in the sense:

$$\lim_{\boldsymbol{h} \to 0} \frac{f(\boldsymbol{a}+\boldsymbol{h}) - P_{f,\boldsymbol{a}}^k(\boldsymbol{a}+\boldsymbol{h})}{|\boldsymbol{h}|^k} = 0$$

Definition *Little-oh*

Let $U \subseteq \mathbb{R}^n$ be a neighborhood of 0, and let $f, h : U \setminus \{0\} \to \mathbb{R}$ be two functions with $h > 0$.
We say that f is **little-oh** of h, denoted $f \in o(h)$, if:

$$\lim_{\boldsymbol{x} \to 0} \frac{f(\boldsymbol{x})}{h(\boldsymbol{x})} = 0$$

Theorem *Chain rule for Taylor polynomials*

Let $U \subseteq \mathbb{R}^n$ be open and $g : U \to V$ and $f : V \to \mathbb{R}$ be of class C^k. Then $f \circ g$ is of class C^k, and if $g(\boldsymbol{a}) = \boldsymbol{b}$, the Taylor polynomial $P_{f \circ g,\boldsymbol{a}}^k(\boldsymbol{a}+\boldsymbol{h})$ is obtained by considering the polynomial

$$\boldsymbol{h} \mapsto P_{f,b}^k(P_{g,\boldsymbol{a}}^k(\boldsymbol{a}+\boldsymbol{h})$$

and discarding the terms of degree $> k$.

Theorem *Taylor polynomial for implicit functions*

Let F be a function of class C^k with $k \geq 1$ such that $F(a,b) = 0$.
Then, the implicit function is also C^k, and its degree k Taylor polynomial, $P^k_{g,b} : \mathbb{R}^m \to \mathbb{R}^n$ satisfies

$$P^k_{F,(a,b)}(P^k_{g,b}(b+h), b+h) = 0$$

1.6.3 Quadratic forms

Definition *Quadratic form*

A **quadratic form** $Q : \mathbb{R}^n \to \mathbb{R}$ is a polynomial function in variables $x_1, \dots, x_n$ all of whose terms are of degree 2.
Alternatively, a n-variable quadratic form is a polynomial of the form

$$Q(\boldsymbol{x}) = \sum_{I \in \mathcal{I}^2_n} a_I \cdot \boldsymbol{x}^I \qquad a_I \in \mathbb{R}$$

Theorem

For any quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$, there exist $m = k + l$ linearly independent functions $\alpha_1, \dots, a_m : \mathbb{R}^n \to \mathbb{R}$ such that:

$$Q(\boldsymbol{x}) = (\alpha_1(\boldsymbol{x}) + \dots + \alpha_k(\boldsymbol{x}))^2 - (\alpha_{k+1}(\boldsymbol{x}) + \dots + \alpha_{k+l}(\boldsymbol{x}))$$

The number k of plus signs and the number l of minus signs are independent of the choice of α's.

Definition

The **signature** of a quadratic form is the pair (k, l) (from the previous theorem)

Proposition

The following two sets are isomorphic:

$$\{\text{Quadratic forms in } n \text{ variables}\} \cong \{A \in \mathbb{R}^{n \times n} : A^\top = A\}$$

Definition *Equivalent quadratic forms*

Two quadratic forms Q, Q' in n variables are called **equivalent** if there exists a nonsingular matrix $S \in \mathbb{R}^{n \times n}$ such that $Q'(\boldsymbol{x}) = Q(S\boldsymbol{x})$

Proposition

Suppose Q, Q' are quadratic forms with matrices A, A' respectively.

$$Q, Q' \text{ are equivalent by a matrix } S \iff A' = S^\top A S$$

Definition *Definite quadratic form*

- A quadratic form Q is positive definite if $Q(\boldsymbol{x}) > 0$ whenever $\boldsymbol{x} \neq 0$, or alternatively if the signature is $(n, 0)$.
- A quadratic form Q is negative definite if $Q(\boldsymbol{x}) < 0$ whenever $\boldsymbol{x} \neq 0$, or alternatively if the signature is $(0, n)$.

Proposition

Let Q be a quadratic form with signature (k, l).

- k is the largest dimension of a subspace on $\mathbb{R}^n$ on which Q is positive definite.
- l is the largest dimension of a subspace of $\mathbb{R}^n$ on which Q is negative definite.
- The signature of Q is independent of its representation, i.e., is independent of coordinates.

Definition *Rank and degeneracy*

The **rank** of a quadratic form is the number of linearly independent squares in its representation as a sum of squares.
A quadratic form on $\mathbb{R}^n$ is **non-degenerate** if its rank is n. It is **degenerate** if its rank is less than n.

Proposition

Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a positive definite quadratic form. Then there exists $C > 0$ such that $Q(\boldsymbol{x}) \geq C|\boldsymbol{x}|^2$.

1.6.4 Critical points

Definition *Critical point and value*

Let U be open and $f : U \to R$ differentiable.
A **critical point** of f is a point $u \in U$ where the derivative of f vanishes.
The value of f at the critical point is called a **critical value**.

Theorem

Let U be open and $f : U \to \mathbb{R}$ differentiable. If $x_0 \in U$ is a local minimum or a local maximum, then $Df(x_0) = 0$.

Definition *Signature of a critical point*

Let U be open and $f : U \to \mathbb{R}$ twice differentiable. Let x_0 be a critical point of f.
The **signature** of the critical point x_0 is the signature of the quadratic form

$$Q_{f,x_0}(h) := \sum_{I \in \mathcal{I}_n^2} \frac{D_I f(x_0)}{I!} h^I$$

Equivalently, if we define the **Hessian matrix** $H(x)$ by $H_{ij}(x) = D_i D_j f(x)$, then

$$Q_{f,x_0}(h) = \frac{1}{2}(h^\top H(x_0) h)$$

Theorem *Minima, maxima and saddles*

Let U be open and $f : U \to \mathbb{R}$ of class C^2. Let x_0 be a critical point of f.

1. If the signature of x_0 is $(n, 0)$, i.e. Q_{f,x_0} is positive definite, then x_0 is a **strict local minimum**.
2. If the signature of x_0 is $(0, n)$, i.e. Q_{f,x_0} is negative definite, then x_0 is a **strict local maximum**.
3. If the signature of x_0 is (k, l) with $k, l > 0$, then x_0 is neither a local minimum nor a local maximum. In this case we call x_0 a **saddle**. A saddle can be degenerate or non-degenerate.

2 Integration

2.1 Definitions

Definition *Integration over a region*

$$\int_A g(x) |\mathrm{d}^n x| = \int_{\mathbb{R}^n} g(x) \mathbf{1}_A |\mathrm{d}^n x|$$

Definition *Support*

The **support** $\mathrm{Supp}(f)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the closure of the set $\{x \in \mathbb{R}^n \mid f(x) \neq 0\}$
$f(x)$ has **bounded support** if there exists $R > 0$ such that $f(x) = 0$ for some $x > R$.

Notation

$$M_A(f) = \sup_{x \in A} f(x) \qquad m_A(f) = \inf_{x \in A} f(x)$$

Definition *Oscillation*

The **oscillation** $\mathrm{osc}_A(f)$ of f over A is $M_A(f) - m_A(f)$.

2.1.1 Dyadic cubes

Definition *Dyadic cube and paving*

A **dyadic cube** $C_{k,N} \subseteq \mathbb{R}^n$, where $k = (k_1, \dots, k_n)$ is a vector of integers, is given by

$$C_{k,N} = \left\{ x \in \mathbb{R}^n \,\middle|\, \frac{k_i}{2^N} < x_i < \frac{k_i + 1}{2^N}, 1 \leq i \leq n \right\}$$

The collection of all cubes $C_{k,N}$ at a single level N, denoted by $\mathcal{D}_N(\mathbb{R}^n)$, is the N-th **dyadic paving** of $\mathbb{R}^n$

Proposition *Volume of a dyadic cube*

The n-dimensional volume of a cube $C \in \mathcal{D}_n(\mathbb{R}^n)$ is $\mathrm{vol}_n\, C = \left(\frac{1}{2^N}\right)^n = \frac{1}{2^{Nn}}$

The distance between two points in the same cube is bounded by $\frac{\sqrt{n}}{2^N}$

Definition *Upper and lower sum*

Let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded with bounded support. The N-th **upper and lower sums** of a function f are

$$U_n(f) = \sum_{C \in \mathcal{D}_N} M_C(f)\, \mathrm{vol}_n\, C = \frac{1}{2^{Nn}} \sum_{C \in \mathcal{D}_N} M_C(f)$$

$$L_n(f) = \sum_{C \in \mathcal{D}_N} m_C(f)\, \mathrm{vol}_n\, C = \frac{1}{2^{Nn}} \sum_{C \in \mathcal{D}_N} m_C(f)$$

2.1.2 Integrable functions

Definition *Upper and lower integrals*

$$U(f) = \lim_{N \to \infty} U_N(f) \qquad L(f) = \lim_{N \to \infty} L_N(f)$$

Definition *Integrable function*

A function $f : \mathbb{R}^n \to \mathbb{R}$, bounded with bounded support, is **integrable** if $U(f) = L(f)$. Its **integral** is then

$$\int_{\mathbb{R}^n} f |\mathrm{d}^n x| = U(f) = L(f)$$

2.2 Volume of sets

Definition *n-dimensional volume*

Let $A \subseteq \mathbb{R}^n$ be a bounded set. If $\mathbf{1}_A$ is integrable, then the n-**dimensional volume** of A is

$$\mathrm{vol}_n\, A := \int_{\mathbb{R}^n} |\,\mathrm{d}^n x| 1_A |\,\mathrm{d}^n x|$$

Riemann sum

Choose any $x_{k,N} \in C_{k,N}$. The **Riemann sum**

$$R(f,N) = \sum_{k \in \mathbb{Z}^n} \mathrm{vol}_n(C_{k,n}) f(x_{k,N})$$

converges to the integral as $N \to \infty$

2.2.1 Pavable sets

Definition *Pavable set*

A set is **pavable** if it has well-defined volume (the indicator function is integrable over the set)

Proposition

- The n-dimensional parallelogram (a product of intervals) has volume equal to the product of its side lengths.
- If two disjoint sets $A, B \in \mathbb{R}^n$ are pavable, then so is their union, and $\mathrm{vol}_n(A \cup B) = \mathrm{vol}_n(A) + \mathrm{vol}_n(B)$
- Let $A \subseteq \mathbb{R}^n$ be pavable and $v \in \mathbb{R}^n$ a vector. Then the shift $A + v$ is pavable and has the same volume as A.
- If $A \subseteq \mathbb{R}^n$ has volume, and $t \in \mathbb{R}$, then tA has volume, and $\mathrm{vol}_n(tA) = |t|^n \mathrm{vol}_n(A)$

2.2.2 Volume zero and integrability conditions

Proposition

A bounded set $X \subseteq \mathbb{R}^n$ has volume 0 if and only if for every $\varepsilon > 0$ there exists N such that

$$\sum_{\substack{C \in \mathcal{D}_N(\mathbb{R}^n) \\ C \cap X \neq 0}} \mathrm{vol}_n(C) \leq \varepsilon$$

Theorem

A function $f : \mathbb{R}^n \to \mathbb{R}$ is integrable if and only if:

- it is bounded with bounded support
- for all $\varepsilon > 0$ there exists N such that $\sum_{C \in A} \mathrm{vol}_v(C) < \varepsilon$ where $A = \{C \in \mathcal{D}_N(\mathbb{R}^n) \mid \mathrm{osc}_C(f) > \varepsilon\}$

Proposition

If $M \subseteq \mathbb{R}^n$ is a manifold of dimension $k < n$, then any compact subset $X \subseteq M$ satisfies $\mathrm{vol}_n(X) = 0$.

Theorem

Any continuous function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support is integrable.

Corollary

Let $X \subseteq \mathbb{R}^n$ be compact and let $f : X \to \mathbb{R}$ be continuous.
Then the graph of f has $(n+1)$-dimensional volume 0.

Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}$ be continuous.
Then any compact part Y of the graph of f has $(n+1)$-dimensional volume 0.

Theorem

A function $f : \mathbb{R}^n \to \mathbb{R}$, bounded with bounded support, is integrable if it is continuous except on a set of volume 0.

2.2.3 Measure zero

Definition *Measure zero*

A set $X \subseteq \mathbb{R}^n$ has **measure 0** if there exists an infinite sequence of open boxes:

$$B_i := \{x \in \mathbb{R}^n \mid a_i < x_i < a_i + \delta \quad (i = 1, \dots, n)\}$$

such that:

$$X \subseteq \bigcup_{i \in \mathbb{N}} B_i \quad \text{and} \quad \sum_{i \in \mathbb{N}} \mathrm{vol}_n(B_i) \leq \varepsilon$$

By "**almost everywhere**" we mean everywhere except on a set of measure zero.

Proposition

A set with volume 0 has measure 0. The converse is not necessarily true.

Theorem

A countable union of sets of measure 0 has measure 0.

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be bounded with bounded support.

$$f \text{ is integrable} \iff f \text{ is continuous except on a set of measure 0}$$

2.3 Integration rules

Proposition *Rules for computing integrals*

Assume that $f, g : \mathbb{R}^n \to \mathbb{R}$ are integrable. The following right hand sides are integrable:

1. $\displaystyle\int_{\mathbb{R}^n} |(f+g)||\mathrm{d}^n x| = \int_{\mathbb{R}^n} f|\mathrm{d}^n x| + \int_{\mathbb{R}^n} g|\mathrm{d}^n x|$
2. $\displaystyle\int_{\mathbb{R}^n} (af)|\mathrm{d}^n x| = a\int_{\mathbb{R}^n} f|\mathrm{d}^n x|$
3. If $f \leq g$, then $\displaystyle\int_{\mathbb{R}^n} f|\mathrm{d}^n x| \leq \int_{\mathbb{R}^n} g|\mathrm{d}^n x|$
4. $\displaystyle\left|\int_{\mathbb{R}^n} f|\mathrm{d}^n x|\right| \leq \int_{\mathbb{R}^n} |f||\mathrm{d}^n x|$

Proposition

Let $f_1(x)$ be integrable on $\mathbb{R}^n$ and $f_2(y)$ be integrable on $\mathbb{R}^m$.
Then the function $g(x, y) = f1(x)f2(y)$ on $\mathbb{R}^{n+m}$ is integrable and

$$\int_{\mathbb{R}^{n+m}} g|\mathrm{d}^n x||\mathrm{d}^m y| = \left(\int_{\mathbb{R}^n} f_1|\mathrm{d}^n x|\right)\left(\int_{\mathbb{R}^m} f_2|\mathrm{d}^m y|\right)$$

2.3.1 Fubini's theorem

Theorem *Fubini's theorem*

Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be an integrable function, and suppose that for each $x \in \mathbb{R}^n$, the function $y \mapsto f(x, y)$ is integrable. Then the function

$$x \mapsto \int_{\mathbb{R}^m} f(x, y) |\,\mathrm{d}^m y|$$

is integrable and

$$\int_{\mathbb{R}^{n+m}} f(x, y) |\,\mathrm{d}^n x||\,\mathrm{d}^m y| = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} r(x, y) |\,\mathrm{d}^m y| \right) |\,\mathrm{d}^n x|$$

2.4 Volume of k-dimensional objects

2.4.1 Volume of linear transformations

Definition *Paralellogram and cube*

Let $v_1, \ldots, v_k$ be vectors in $\mathbb{R}^n$. The k-**paralellogram** spanned by $v_1, \ldots, v_k$ is

$$P(v_1, \ldots, v_k) := \{t_1 v_1 + \ldots + t_k v_k \mid 0 \leq t_i \leq 1\}$$

If the vectors $v_1, \ldots, v_k$ are the standard basis vectors of $\mathbb{R}^k$, we call it a k-dimensional **unit cube**.

Lemma

1. Let C be a dyadic cube, T a linear transformation and Q a unit cube.

$$\mathrm{vol}_n\, T(C) = \mathrm{vol}_n\, T(Q)\, \mathrm{vol}_n\, C$$

2. Let $A \subseteq \mathbb{R}^n$ be a (pavable) set, T a linear transformation and Q a unit cube.

$$\mathrm{vol}_n\, T(A) = \mathrm{vol}_n\, T(Q)\, \mathrm{vol}_n\, A$$

3. Let Q be a unit cube and T a linear transformation.

$$\mathrm{vol}_n\, T(Q) = |\det T|$$

Theorem

Let T be a linear transformation and denote by $[T]$ its associated matrix.
Then, for any pavable set $A \subseteq \mathbb{R}^n$, the image $T(A)$ is pavable, and

$$\mathrm{vol}_n\, T(A) = |\det[T]|\, \mathrm{vol}_n\, A$$

Proposition

Let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^n$. Then

$$\mathrm{vol}_n\, P(v_1, \ldots, v_n) = \det[v_1 \ \ldots \ v_n]$$

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation, and $f : \mathbb{R}^n \to \mathbb{R}$ integrable. Then $f \circ T$ is integrable, and

$$\int_{\mathbb{R}^n} f(y) |\,\mathrm{d}^n y| = |\det T| \int_{\mathbb{R}^n} f(T(x)) |\,\mathrm{d}^n x|$$

2.4.2 Change of coordinates

Theorem *Change of coordinates*

Let X be a compact subset of $\mathbb{R}^n$ with boundary ∂X of volume 0. Let U be an open set containing X.
Let $\phi : U \to \mathbb{R}^n$ be a C^1-mapping such that:

- ϕ is injective on $X \setminus \partial X$
- The derivative of ϕ is satisfies the Lipschitz condition
- $[D\phi(x)]$ is invertible at every $X \in X \setminus \partial X$.

Set $Y = \phi(X)$. Then, if $f : Y \to \mathbb{R}$ is integrable, $|\det[D\phi]|(f \circ \phi)$ is integrable on X and:

$$\int_Y f(y)|\,\mathrm{d}^n y| = \int_X |\det[D\phi(x)]| \cdot (f \circ \phi)|\,\mathrm{d}^n x|$$

2.4.3 Volume of manifolds

Definition *Volume of a paralellogram*

Let $T = [v_1, \dots, v_k]$ be an $n \times k$ real matrix. Then the k-dimensional volume of $P(v_1, \dots, v_k)$ is

$$\operatorname{vol}_k P(v_1, \dots, v_k) := \sqrt{\det(T^\top T)}$$

This is also true if the parallelogram is **anchored** at $x \in \mathbb{R}^n$.

$$\operatorname{vol}_k P_x(v_1, \dots, v_k) = \operatorname{vol}_k P(v_1, \dots, v_k)$$

Definition *k-dimensional volume 0*

A bounded subset $X \subseteq \mathbb{R}^n$ has **k-dimensional volume 0** if

$$\lim_{N \to \infty} \sum_{\substack{C \in \mathcal{D}_N(\mathbb{R}^n) \\ C \cap X \neq 0}} \left(\frac{1}{2^N}^k\right) = 0$$

An arbitrary subset $X \subseteq \mathbb{R}^n$ has k-dimensional volume 0 if

$$\operatorname{vol}_k(X \cap B_R(0)) = 0 \qquad \text{for all } R$$

Proposition

Let m, k, n be integers satisfying $0 \leq m < k \leq n$.
If $M \subseteq \mathbb{R}^n$ is an m-dimensional manifold, then any closed subset $X \subset M$ has k-dimensional volume 0.

Definition *Parametrization of a manifold*

Let $M \subseteq \mathbb{R}^n$ be a k-dimensional manifold and let $U \subseteq \mathbb{R}^k$ be a subset with boundary of k-dimensional volume 0.
Let $X \subseteq U$ be such that $U \setminus X$ is open. Then a continuous mapping $\gamma : U \to \mathbb{R}^n$ parametrizes M if:

1. $M \subseteq \gamma(U)$
2. $\gamma(U \setminus X) \subseteq M$
3. $\gamma : U \setminus X \to M$ is injective and C^1
4. the derivative $D\gamma(u)$ is injective for all $u \in U \setminus X$
5. X has k-dimensional volume 0
6. $\gamma(X) \cap C$ has k-dimensional volume 0 for every compact subset $C \subseteq M$

Theorem

All manifolds can be parametrized.

Definition *Volume of a k-dimensional manifold*

Let $M \subseteq \mathbb{R}^n$ be a smooth k-dimensional manifold, U a pavable subset of $\mathbb{R}^k$, and $\gamma : U \to M$ a parametrization. Let X be as in the definition of parametrization. Then

$$\operatorname{vol}_n M = \int_{\gamma(U \setminus X)} |\,\mathrm{d}^k x| = \int_{U \setminus X} \underbrace{\left(|\,\mathrm{d}^k x| (\underbrace{P_{\gamma(u)}(D_1\gamma(u), \dots, D_k\gamma(u))}_{\text{Parallelogram anchored at } \gamma(u)}) \right)}_{\text{Volume of the parallelogram}} |\,\mathrm{d}^k u|$$

$$= \int_{U \setminus X} \sqrt{\det([D\gamma(u)]^\top [D\gamma(u)])}\, |\,\mathrm{d}^k u|$$

Definition *Integral with respect to volume*

Let $M \subseteq \mathbb{R}^n$ be a smooth k-dimensional manifold, U a pavable subset of $\mathbb{R}^k$, and $\gamma : U \to M$ a parametrization. Then $f : M \to \mathbb{R}$ is **integrable over M with respect to volume** if the following integral exists:

$$\int_M f(x) |\,\mathrm{d}^k x| := \int_{U \setminus X} f(\gamma(u)) \sqrt{\det([D\gamma(u)]^\top [D\gamma(u)])} |\,\mathrm{d}^k u|$$

In particular, if $Y \subseteq M$ is a subset such that $\mathbf{1}_{\gamma^{-1}(Y)}$ is integrable, then

$$\operatorname{vol}_k Y = \int_{U \setminus X} \mathbf{1}_Y(\gamma(u)) \sqrt{\det([D\gamma(u)]^\top [D\gamma(u)])} |\,\mathrm{d}^k u|$$

Proposition

The integral (both its existence and its value)

$$\int_{U \setminus X} f(\gamma(u)) \sqrt{\det([D\gamma(u)]^\top [D\gamma(u)])} |\,\mathrm{d}^k u|$$

as in the definition above is independent of the choice of parametrization.

3 Forms and orientation

3.1 Constant k-forms

3.1.1 Permutations

Definition *Permutation*

A **permutation** of a set S is a bijection $S \to S$ that re-orders its elements.

Definition *Sign of a permutation*

An **inversion** is a pair of elements (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$.
The **sign** of a permutation is defined by:

$$\operatorname{sgn}(\sigma) = (-1)^{N(\sigma)}$$

where $N(\sigma)$ is the number of inversions in the permutation.

Alternatively, the sign is 1 (**even**) if the permutation can be obtained by an even number of pairwise swaps, and the sign is -1 (**odd**) if the permutation can be obtained by an odd number of pairwise swaps.

3.1.2 Elementary constant k-forms

Definition *Antisymmetric function*

A function is $\phi : (\mathbb{R}^n)^k \to \mathbb{R}$ is **antisymmetric** if for any permutation σ of the indices $\{1, 2, \ldots, k\}$:

$$\phi(v_1, v_2, \ldots, v_k) = \operatorname{sgn}(\sigma) \cdot \phi(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)})$$

Swapping two arguments of an antisymmetric function changes the sign of the function.

Definition *Multilinear function*

A function $\phi : (\mathbb{R}^n)^k \to \mathbb{R}$ is **multilinear** if it is linear in all of its arguments:

$$\phi(av_1 + bw_1, v_2, \ldots, v_k) = a\phi(v_1, v_2, \ldots, v_k) + b\phi(w_1, v_2, \ldots, v_k) \qquad \text{for all} \quad v_1, w_1, v_2, \ldots, v_k \in \mathbb{R}^n \quad a, b \in \mathbb{R}$$

Definition *Constant k-form*

A **constant k-form** on $\mathbb{R}^n$ is a function $\phi : (\mathbb{R}^n)^k \to \mathbb{R}$ that takes k vectors and returns a number $\phi(v_1, \ldots, v_k)$ such that ϕ is multilinear and antisymmetric as a function of the vectors.
The number k is called the **degree** of the form.

Definition *Elementary constant k-form*

An **elementary constant k-form** is an expression of the form:

$$\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2} \wedge \ldots \wedge \mathrm{d}x_{i_k} \qquad (1 \leq i_1 < \ldots < i_k \leq n)$$

Evaluated on the vectors $v_1, \ldots, v_k$ it gives the determinant of the $k \times k$ matrix obtained by selecting rows $i_1, \ldots, i_k$ of the matrix $[v_1, \ldots, v_k]$. The only **elementary 0-form** is the form denoted 1, which maps the empty set to 1.

Definition *Linear combinations of forms*

Let ϕ and ψ be two k-forms, and a a scalar.

$$(\phi + \psi)(v_1, \ldots, v_k) := \phi(v_1, \ldots, v_k) + \psi(v_1, \ldots, v_k)$$
$$(a\phi)(v_1, \ldots, v_k) := a\,(\phi(v_1, \ldots, v_k))$$

The **space of constant k-forms** is a vector space, denoted by $A_c^k(\mathbb{R}^n)$.

Theorem

The elementary constant k-forms form a basis of $A_c^k(\mathbb{R}^n)$.

Theorem

The space $A_c^k(\mathbb{R}^n)$ has dimension $\displaystyle\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Definition *Wedge product*

The **wedge product** of the forms $\phi \in A_n^k(\mathbb{R}^n)$ and $\psi \in A_c^\ell(\mathbb{R}^n)$ is the element $\phi \wedge \psi \in A^{k+\ell}(\mathbb{R}^n)$ defined by

$$(\phi \wedge \psi)(v_1, v_2, \ldots, v_{k+\ell}) := \sum_{\sigma \in \operatorname{Perm}(k,\ell)} \operatorname{sgn}(\sigma)\phi\big(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\big)\, \psi\big(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\big)$$

where $\operatorname{Perm}(k, \ell)$ is the set of permutations σ of the numbers $1, 2, \ldots, k + \ell$ such that

$$\sigma(1) < \sigma(2) < \ldots < \sigma(k) \qquad \sigma(k+1) < \sigma(k+2) < \ldots < \sigma(k+\ell)$$

Proposition *Properties of the wedge product*

1. (**Distributivity**) $\phi \wedge (\psi_1 + \psi_2) = \phi \wedge \psi_1 + \phi \wedge \psi_2$
2. (**Associativity**) $(\phi_1 \wedge \phi_2) \wedge \phi_3 = \phi_1 \wedge (\phi_2 \wedge \phi_3)$
3. (**Skew commutativity**) If ψ is a k-form and ϕ is an ℓ-form, then $\psi \wedge \phi = (-1)^{kl}\phi \wedge \psi$

3.2 Differential forms

Definition *k-form field*

A **k-form field** (or **differential form**) on an open subset $U \subseteq \mathbb{R}^n$ is a map $\phi : U \to A_c^k(\mathbb{R}^n)$.
The **space of k-form fields** is denoted by $A^k(U)$.

Differential forms

A differential form is a function that maps k-dimensional parallelograms anchored at points in U and returns numbers given by

$$\phi(P_x(v_1, \dots, v_k)) := \phi(x)(v_1, \dots, v_k)$$

where $\phi(x)$ is of the form

$$\phi(x) = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1 \dots i_k}(x)\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$$

and $a_{i_1, \dots, i_k}$ are real-valued functions of $x \in U$.
We say a differential form is of class C^p if $a_{i_1, \dots, i_k}$ are of class C^p.

3.2.1 Integrals of differential forms over parametrized domains

Definition *Parametrized domain*

Let $U \subseteq \mathbb{R}^k$ be a bounded open set, with boundary of k-dimensional volume 0.
A C^1-mapping $\gamma : U \to \mathbb{R}^n$ defines a **domain in $\mathbb{R}^n$ parametrized by U**. We denote the pair (U, γ) by $[\gamma(U)]$.

Definition *Integral of a differential form over a parametrized domain*

Let $U \subseteq \mathbb{R}^k$ be a bounded open set with boundary of k-dimensional volume 0.
Let $V \subseteq \mathbb{R}^n$ be open and let $[\gamma(U)]$ be a parametrized domain in V. Let φ be a k-form field on V.
Then the integral of φ over $[\gamma(U)]$ is

$$\int_{\gamma(U)} \varphi := \int_U \varphi\left(P_{\gamma(u)}(D_1\gamma(\mathbf{u}), \dots, D_k\gamma(\mathbf{u}))\right) |\mathrm{d}^k \mathbf{u}|$$

3.3 Orientation of manifolds

Definition *Orientation of a vector space*

Let V be a finite-dimensional real vector space, and let $\mathcal{B}_V$ be the set of bases of V. An **orientation** of V is a map $\Omega : \mathcal{B}_V \to \{+1, -1\}$ such that if $\{v\}$ and $\{w\}$ are two bases with change of basis matrix $[P_{w \to v}]$, then

$$\Omega(\{w\}) = \operatorname{sgn}(\det[P_{w \to v}])\Omega(\{v\})$$

A basis $\{u\} \in \mathcal{B}_V$ is called **direct** if $\Omega(\{w\}) = +1$, and **indirect** if $\Omega(\{w\}) = -1$
To orient V, we choose a basis of V and declare it to be direct.
The orientation for which $\{v\}$ is direct is denoted $\Omega^{\{v\}}$ and is called the **orientation specified by** $\{v\}$.
The **standard orientation** on $\mathbb{R}^n$, denoted Ω^{st}, is defined by declaring the standard basis to be direct.
Subspaces of $\mathbb{R}^n$ do not necessarily have a standard orientation.

Definition *Orientation of a manifold*

Let $M \subseteq \mathbb{R}^n$ be a k-dimensional manifold. Define:

$$\mathcal{B}(M) := \{(\boldsymbol{x}, \boldsymbol{v}_1, \dots, \boldsymbol{v}_k) \in \mathbb{R}^n \times (\mathbb{R}^n)^k\}$$

where $\boldsymbol{x} \in M$ and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ is a basis of the tangent space $T_x M$.
Let $\mathcal{B}_x(M) \subseteq \mathcal{B}(M)$ be the subset where the first coordinate is $\boldsymbol{x}$, that is $\mathcal{B}_x(M) = \{\boldsymbol{x}\} \times \mathcal{B}_{T_x M}$.

An **orientation of a manifold** $M \subseteq \mathbb{R}^n$ is a continuous map $\Omega : \mathcal{B}(M) \to \{+1, -1\}$ whose restriction Ω_x to each $\mathcal{B}_x(M)$ is an orientation of $T_x M$.

Examples of orientations

- To orient a discrete set of points (a 0-dimensional manifold on $\mathbb{R}^n$) we simply assign $+1$ or -1 to each point.
- An n-dimensional open subset U of $\mathbb{R}^n$ carries the standard orientation of $\mathbb{R}^n$.
- Let $C \subset \mathbb{R}^n$ be a smooth curve (a 1-dimensional manifold in $\mathbb{R}^n$).
  Let f be a non-vanishing tangent vector field that varies continuously with x, i.e. a continuous map $f : x \mapsto f(x) \in T_x M$. Then for every basis v of $T_x C$, f defines an orientation of C by the formula

$$\Omega_x^f(x, v) := \operatorname{sgn}(f(x) \cdot v)$$

Orientation by transverse vector field

Let $S \subseteq \mathbb{R}^3$ be a smooth surface and let $n : S \to \mathbb{R}^3$ be a continuous **transverse vector field** n on S, that is a vector field defined at every $x \in S$ such that $n(x) \neq 0$ and $n(x) \notin T_x S$.

Then, one can define an **orientation by transverse vector field** n, denoted Ω^n, of S by

$$\Omega^n(v_1, v_2) := \operatorname{sgn}(\det[n(x), v_1, v_2])$$

for all $x \in S$ and all bases v_1, v_2 of $T_x S$.

Proposition

Let $U \subseteq \mathbb{R}^n$ be open and let $f : U \to \mathbb{R}^{n-k}$ be a C^1 map such that $Df(x)$ is surjective at all $x \in M := f^{-1}(x)$. Then the map

$$\Omega(v_1, \ldots, v_n) := \operatorname{sgn}\det[\nabla f_1(x), \ldots, \nabla f_{n-k}, v_1, \ldots, v_k$$

is an orientation of M.

Definition *Orienting-preserving linear transformation*

Let V be a k-dimensional vector space oriented by $\Omega : \mathcal{B}(V) \to \{+1, -1\}$.
A linear transformation $T : \mathbb{R}^k \to V$ is:

- **orientation-preserving** if $\Omega(T(e_1), \ldots, T(e_k)) = +1$
- **orientation-reversing** if $\Omega(T(e_1), \ldots, T(e_k)) = -1$

Definition *Orientation-preserving parametrization of a manifold*

Let $M \subseteq \mathbb{R}^m$ be a k-dimensional manifold oriented by Ω, and let $U \subseteq \mathbb{R}^k$ be a subset with boundary of k-dimensional volume 0. Let $\gamma : U \to \mathbb{R}^m$ parametrize M, with the set X as in the definition of parametrization.
Then γ is **orientation-preserving** if:

$$\Omega(D_1\gamma(u), \ldots, D_k\gamma(u)) = +1 \qquad \text{for all } u \in U \setminus X$$

Proposition

Let M be an oriented manifold, and $\gamma : U \to M$ a parametrization of an open subset of M, with $U \setminus X$ connected. Then if γ preserves orientation at a single point of U, it preserves orientation at every point of U.

Theorem

Let $M \subseteq \mathbb{R}^n$ be a k-dimensional manifold, let U_1, U_2 be open subsets of $\mathbb{R}^k$, and let $\gamma_1 : U_1 \to \mathbb{R}^n$ and $\gamma_2 : U_2 \to \mathbb{R}^n$ be two orientation-preserving parametrizations of M. Then for any k-form φ defined on a neighborhood of M,

$$\int_{[\gamma_1(U_1)]} \varphi = \int_{[\gamma_2(U_1)]} \varphi$$

Definition

Let $M \subseteq \mathbb{R}^n$ be a k-dimensional oriented manifold, φ a k-form field on a neighborhood of M, and $\gamma : U \to M$ any orientation-preserving parametrization of M. Then

$$\int_M \phi = \int_{[\gamma(U)]} \phi = \int_U \varphi \left(P_{\gamma(u)}(D_1\gamma(u), \ldots, D_k\gamma(u))\right) |\mathrm{d}^k u|$$

3.4 Forms on $\mathbb{R}^3$

0-forms

A 0-form is simply a number and a 0-form field is simply a function.
The rule $f(P_x) = f(x)$ makes a function $f : U \to \mathbb{R}$ into a 0-form field.

Definition *1-forms and the work form of a vector field*

The **work form** $W_{\vec{F}}$ of a vector field $\vec{F}$ is the 1-form field defined by:

$$W_{\vec{F}}(P_x(v)) := \vec{F}(x) \cdot v = F_1 \,\mathrm{d}x_1 + \ldots + F_n \,\mathrm{d}x_n$$

Definition *2-forms and the flux form of a vector field*

The **flux form** $\Phi_{\vec{F}}$ of a vector field $\vec{F}$ is the 2-form field defined by:

$$\Phi_{\vec{F}}(P_x(v, w)) := \det[\vec{F}(x), v, w] = F_1 \,\mathrm{d}y \wedge \mathrm{d}z - F_2 \,\mathrm{d}x \wedge \mathrm{d}z + F_3 \,\mathrm{d}x \wedge \mathrm{d}y$$

Definition *3-forms and the mass form of a function*

Let U be a subset of $\mathbb{R}^3$ and $f : U \to \mathbb{R}$ a function. The **mass form** M_f is the 3-form defined by

$$M_f\,(P_x(v_1, v_2, v_3)) := f(x) \underbrace{\det[v_1, v_2, v_3]}_{\text{signed volume of } P}$$

Integrals of work, flux and mass forms

The **work** of a vector field $\vec{F}$ along an oriented curve C is $\displaystyle\int_C W_{\vec{F}} = \int_a^b \vec{F}(\gamma(u)) \cdot \gamma'(u) \,\mathrm{d}u$

The **flux** of a vector field $\vec{F}$ along an oriented surface S is $\displaystyle\int_S \Phi_{\vec{F}} = \int_U \det\left[\vec{F}(\gamma(u)), D_1\gamma(u), D_2\gamma(u)\right] |\mathrm{d}^2 u|$

The integral of a mass form M_f over V, simply called the integral of f, is $\displaystyle\int_V M_f = \int_U f(\gamma(u)) \det[D\gamma(u)] \,|\mathrm{d}^3 u|$

In all three cases, we assume γ is an orientation-preserving parametrization.

3.5 Boundary of manifolds

Definition *Boundary of a subset*

Let $M \subseteq \mathbb{R}^n$ be a k-dimensional manifold, and $X \subset M$ a subset. The **boundary** of X in M, denoted by $\partial_M X$, is the set of points $x \in M$ such that every neighborhood of x contains points of X and points of $M \setminus X$.

Definition *Smooth boundary*

Let $M \subseteq \mathbb{R}^n$ be a k-dimensional manifold and $X \subset M$ a subset. A point $x \in \partial_M X$ is a **smooth point** of the boundary of X if there exists a neighborhood $V \subset \mathbb{R}^n$ of x and a single C^1 function $g : V \cap M \to \mathbb{R}$ such that:

1. $g(x) = 0$
2. $X \cap V = \{a \in V \cap M \mid g(a) \geq 0\}$
3. $[Dg(x)] : T_x M \to \mathbb{R}$ is surjective.

The set of smooth points of the boundary of X is the **smooth boundary** of X, denoted $\partial_M^s X$.
The **non-smooth boundary** of X is the part of the boundary of X that is not smooth.

Proposition

The smooth boundary $\partial_M^s X$ is a $(k-1)$-dimensional manifold.

Definition *Corner points*

Let $M \subseteq \mathbb{R}^n$ be a k-dimensional manifold and $X \subset M$ a subset. A point $x \in X$ is a **corner point** of **codimension** m if there exists a neighborhood $V \subset \mathbb{R}^n$ of x and a C^1 function $g : V \cap M \to \mathbb{R}^m$ such that

1. $g(x) = 0$
2. $X \cap V = \{a \in V \cap M \mid g(a) \geq 0\}$
3. $[Dg(x)]$ is surjective.

Definition *Pieces of manifolds*

A **piece-with-boundary** of a k-dimensional manifold M is a compact subset $X \subset M$ such that

1. The set of non-smooth points in $\partial_M X$ has $(k-1)$-dimensional volume 0.
2. The smooth boundary has finite $(k-1)$-dimensional volume.

If every point of the boundary $\partial_M X$ is a corner point, then X is a **piece-with-corners**.

Theorem

If $X \subset M$ is a k-dimensional piece-with-boundary, then X has finite k-dimensional volume.

3.5.1 Boundary orientation

Definition *Inward and outward pointing tangent vector*

Let $M \subseteq \mathbb{R}^n$ be a manifold, $X \subset M$ a piece-with-boundary, x a smooth point in $\partial_M X$, and g the function defining x as a smooth point. Let v be tangent to M at x. Then:

- v **points outward** from X if $[Dg(x)]v < 0$
- v **points inward** from X if $[Dg(x)]v > 0$

Definition *Orientation of a boundary*

Let M be a k-dimensional manifold oriented by Ω, and P a piece-with-boundary of M.
Let x be a point of the smooth boundary $\partial_M^s P$ and let $v_{\text{out}} \in T_x M$ be an outward pointing vector.
Then the function:

$$\Omega^\partial : \mathcal{B}(T_x \partial P) \to \{+1, -1\} \qquad \Omega_x^\partial(v_1, \dots, v_{k-1}) := \Omega_x(v_{\text{out}}, v_1, \dots, v_{k-1})$$

defines an orientation on the smooth boundary $\partial_M^s P$.

Proposition *Oriented boundary of a parallelogram*

Let the k-parallelogram $P_x(v_1, \dots, v_k)$ have the standard orientation.
Then its oriented boundary is given by the following, where a hat over a term indicates that it is omitted:

$$\partial P_x(v_1, \dots, v_k) = \sum_{i=1}^{k} (-1)^{i-1} \left(P_{x+v_i}(v_1, \dots, \hat{v}_i, \dots, v_k) - P_x(v_1, \dots, \hat{v}_i, \dots, v_k) \right)$$

4 Exterior derivatives

Definition *Exterior derivative*

Let $U \subseteq \mathbb{R}^n$ be an open subset. The **exterior derivative** $\mathbf{d} : A^k(U) \to A^{k+1}(U)$ is defined by the formula

$$\mathbf{d}\varphi(P_x(v_1, \dots, v_{k+1})) := \lim_{h \to 0} \frac{1}{h^{k+1}} \int_{\partial P_x(hv_1, \dots, hv_{k+1})} \varphi$$

Theorem *Properties of exterior derivatives*

Let φ be a k-form of class C^2 on an open subset $U \subseteq \mathbb{R}^n$:

$$\varphi = \sum_{i \leq i_1 < \cdots < i_k \leq n} a_{i_1,\ldots,i_k} \, \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$$

1. The limit $\lim_{h \to 0} \frac{1}{h^{k+1}} \int_{\partial P_x(h\vec{v}_1,\ldots,h\vec{v}_{k+1})} \varphi$ exists and defines a $(k+1)$-form.
2. The exterior derivative is linear over $\mathbb{R}$: if φ and ψ are k-forms on $U \subseteq \mathbb{R}^n$ and a, b are numbers, then
$$\mathbf{d}(a\varphi + b\psi) = a\,\mathbf{d}\varphi + b\,\mathbf{d}\psi$$
3. The exterior derivative of a constant form is 0.
4. The exterior derivative of the 0-form f is given by $\mathbf{d}f = [Df] = \sum_{i=1}^{n} (D_i f)\,\mathrm{d}x_i$
5. If $f : U \to \mathbb{R}$ is a C^2 function, then
$$\mathbf{d}(f\,\mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}) = \mathbf{d}f \wedge \mathrm{d}x_{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}$$
6. $\mathbf{d}(\,\mathbf{d}(\varphi)) = 0$

Theorem

If φ is a k-form and ψ is an ℓ-form, then

$$\mathbf{d}(\varphi \wedge \psi) = \mathbf{d}\varphi \wedge \psi + (-1)^k \varphi \wedge \mathbf{d}\psi$$

4.1 Gradient, curl and divergence

Definition *Gradient, curl and divergence*

Let $U \subseteq \mathbb{R}^3$ be an open set, $f : U \to \mathbb{R}$ a C^1 function, and $\vec{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$ a C^1 vector field on U.

$$\operatorname{grad} f = \nabla f = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} f = \begin{bmatrix} D_1 f \\ D_2 f \\ D_3 f \end{bmatrix}$$

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} D_2 F_3 - F_3 F_2 \\ D_3 F_1 - D_1 F_3 \\ D_1 F_2 - D_2 F_1 \end{bmatrix}$$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \cdot \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = D_1 F_1 + D_2 F_2 + D_3 F_3$$

Theorem

Let f be a function on $\mathbb{R}^3$ and let $\vec{F}$ be a vector field.

$$\mathbf{d}f = W_{\operatorname{grad} f} \qquad \mathbf{d}W_{\vec{F}} = \phi_{\operatorname{curl} \vec{F}} \qquad \mathbf{d}\Phi_{\vec{F}} = M_{\operatorname{div} \vec{F}}$$

Definition *Laplacian*

In $\mathbb{R}^3$, the Laplacian of a function f, denoted Δf is

$$\Delta f := (D_1^2 + D_2^2 + D_3^2)(f) = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \cdot \begin{bmatrix} D_1 f \\ D_2 f \\ D_3 f \end{bmatrix} = \operatorname{div} \operatorname{grad} f$$

4.2 Pullback

Definition *Pullback by a linear transformation*

Let V, W be vector spaces, $T : V \to W$ a linear transformation, and φ a constant k-form on W.
The **pullback** by T is the mapping $T^* : A_c^k(W) \to A_c^k(V)$ defined by

$$T^*\varphi(v_1, \dots, v_k) := \varphi(T(v_1), \dots, T(v_k)))$$

The k-form $T^*\varphi$ is called the pullback of φ by T.

Proposition *Computing the pullback by a linear transformation*

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
Denote by $x_1, \dots, x_n$ the coordinates in $\mathbb{R}^n$ and by $y_1, \dots, y_m$ the coordinates in $\mathbb{R}^m$. Then

$$T^*(\,\mathrm{d}y_{j_1} \wedge \cdots \wedge \mathrm{d}y_{j_k}) = \sum_{1 \le j_1 < \cdots < j_k \le n} b_{j_1,\dots,j_k}\,\mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_k}$$

where $b_{j_1,\dots,j_k}$ is the number obtained by taking the matrix of T, selecting its rows $i_1, \dots, i_k$, selecting its columns $j_1, \dots, j_k$, and taking the determinant of the resulting matrix.

Definition *Pullback by a C^1 mapping*

If φ is a k-form field on Y, and $f : X \to Y$ is a C^1 mapping, then $f^* : A^k(Y) \to A^k(X)$ is defined by

$$(f^*\varphi)(P_x(v_1, \dots, v_k)) := \varphi\left(P_{f(x)}([Df(x)]v_1, \dots, [Df(x)]v_k)\right)$$

Proposition *Pullbacks by composition*

If $X \subseteq \mathbb{R}^n, Y \subseteq \mathbb{R}^m$ and $Z \subseteq \mathbb{R}^p$ are open, and φ is a k-form on Z, then

$$(g \circ f)^*\varphi = f^* g^* \varphi$$

which is a k-form on X.

4.2.1 Independence of coordinates

Proposition

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be open, $f : X \to Y$ a C^1 map, and φ and ψ respectively a k-form and ℓ-form on Y. Then

$$f^*\varphi \wedge f^*\psi = f^*(\varphi \wedge \psi) \qquad \mathbf{d} f^*\phi$$

Theorem

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be open, $f : X \to Y$ a C^1 map, and ϕ a k-form field on Y. Then

$$\mathbf{d} f^*\phi = f^*\,\mathbf{d}\phi$$

4.3 Stokes' theorem

Theorem *Stokes' theorem*

Let X be a piece-with-boundary of a k-dimensional oriented smooth manifold M in $\mathbb{R}^n$.
Give the boundary ∂X of X the boundary orientation, and let ϕ be a $(k-1)$-form of class C^2 defined on an open set containing X. Then

$$\int_{\partial X} \phi = \int_X \mathbf{d}\phi$$

Note: ∂X is the smooth boundary of X on M.

4.3.1 Applications of Stokes' theorem

Theorem *Fundamental theorem of calculus*

If f is a C^1 function on a neighborhood of $[a, b]$, then

$$\int_a^b f'(t)\,\mathrm{d}t = f(b) - f(a)$$

Theorem *Green's theorem*

Let S be a bounded region of $\mathbb{R}^2$, bounded by a curve C (or curves C_i), carrying a compatible boundary orientation. Let $\vec{F}$ be a vector field defined on a neighborhood of S. Then

$$\int_S \mathbf{d}W_{\vec{F}} = \int_C W_{\vec{F}} \quad \text{or} \quad \int_S \mathbf{d}W_{\vec{F}} = \sum_i \int_{C_i} W_{\vec{F}}$$

If $\vec{F} = \begin{bmatrix} f \\ g \end{bmatrix}$, then Green's theorem is traditionally written as

$$\int_S (D_1 g - D_2 f)\,\mathrm{d}x\,\mathrm{d}y = \int_C f\,\mathrm{d}x + g\,\mathrm{d}y$$

Theorem *Stokes' theorem in $\mathbb{R}^3$*

Let S be an oriented surface in $\mathbb{R}^3$, bounded by a curve C that is given the boundary orientation. Let ϕ be a 1-form field defined on a neighborhood of S. Then

$$\int_S \mathbf{d}\phi = \int_C \phi$$

Suppose C is the union of disjoint simple closed curves C_i. Let $\vec{N}$ be the normal unit vector field defining the orientation of S, and $\vec{T}$ the unit tangent vector field defining the orientation of the C_i. Then

$$\iint_S (\mathrm{curl}\ \vec{F}(x)) \cdot \vec{N}(x)\,|\mathrm{d}^2 x| = \sum_i \int_{C_i} \vec{F}(x) \cdot \vec{T}(x)\,|\mathrm{d}^1 x|$$

Theorem *Divergence theorem*

Let X be a bounded domain in $\mathbb{R}^3$ with the standard orientation of space, and let its boundary ∂X be a union of surfaces S_i, each oriented by the outward normal. Let ϕ be a 2-form field defined on a neighborhood of X. Then

$$\int_X \mathbf{d}\phi = \sum_i \int_{S_i} \phi$$

Let $\phi = \Phi_{\vec{F}}$ and let $\vec{N}$ be the unit outward-pointing vector field on the S_i. Then the above equation can be rewritten

$$\int_X M_{\mathrm{div}\,\vec{F}} = \iiint_X \mathrm{div}\,\vec{F}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \sum_i \iint_{S_i} \vec{F} \cdot \vec{N}|\,\mathrm{d}^2 x|$$

4.3.2 Poincaré lemma

Definition *Closed and exact forms*

A k-form is **closed** if its exterior derivative is 0.
A k-form ϕ is **exact** if there is a $(k-1)$-form ω such that $\phi = \mathbf{d}\omega$.

Lemma *Poincaré lemma*

Let B be an open ball in $\mathbb{R}^n$. Then any smooth closed k-form ϕ defined on B is exact, for $1 \leq k \leq n$.

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